Order continuity from a topological perspective

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Overview

1. Order convergence in partially ordered sets
   - Order convergence
   - Order continuity
2. On partially ordered abelian group
3. On partially ordered vector spaces

Preliminaries

Definition

Let M be a set. A mapping $\phi : A \to M$, where A is a non-empty and directed partially ordered set is called a **net** (in M). We will use the notation $(x_\alpha)_{\alpha \in A}$, where $x_\alpha := \phi(\alpha)$ instead of $\phi : A \to M$ and state the image of the net by saying, that the net is **in** M.

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Example

Every sequence is a net. $(x_n)_{n \in \mathbb{N}}$ sequence in M corresponds to $\mathbb{N} \to M\colon n \mapsto x_n$.

Definition

$\leq$ *partial order* on P $:\Leftrightarrow$ $P \neq \emptyset$, $\leq$ reflexive, transitive and anti-symmetric binary relation.
The set P is then called a *partially ordered set*.

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Definition

P *lattice* $:\Leftrightarrow$ for every non-empty finite subset of P the infimum and the supremum exist in P.
P *Dedekind complete* $:\Leftrightarrow$ every non-empty set which is bounded above has a supremum, and every non-empty set which is bounded below has an infimum.

Preliminaries

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For a net $(x_\alpha)_{\alpha \in A}$ in P we denote $x_\alpha \downarrow$ if $x_\alpha \leq x_\beta$ whenever $\alpha \geq \beta$. For $x \in P$ we write $x_\alpha \downarrow x$ if $x_\alpha \downarrow$ and $\inf\{x_\alpha;\ \alpha \in A\} = x$. Similarly we define $x_\alpha \uparrow$ and $x_\alpha \uparrow x$.

Order topology

Let P be a partially ordered set.

Definition

$U \subseteq P$ *net catching set* for $x \in P$:$\Leftrightarrow$ for all nets $(\hat{x}_\alpha)_{\alpha \in A}$ and $(\check{x}_\alpha)_{\alpha \in A}$ in P with $\hat{x}_\alpha \uparrow x$ and $\check{x}_\alpha \downarrow x$ there is $\alpha \in A$ such that $[\hat{x}_\alpha, \check{x}_\alpha] \subseteq U$.

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Definition

$O \subseteq P$ *order open* :$\Leftrightarrow$ O net catching set for every $x \in O$.
$C \subseteq P$ *order closed* :$\Leftrightarrow$ $P \setminus C$ order open.

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For every $p \in P$ the set $P_{\geq p}$ is closed.

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$\tau_o(\mathbb{R}^n)$ is the standard topology of $\mathbb{R}^n$, if ordered component wise.

Order topology

Example

Let $(X, ||\cdot||)$ be a reflexive Banach space. Let f be a linear functional on X such that $||f|| := \sup\{\frac{|f(x)|}{||x||}; x \in X\} = 1$ and $\epsilon \in (0,1)$. Then the **ice cream cone** with parameters f and ϵ is the cone

$$K_{f,\epsilon} := \{x \in X;\ f(x) \geq \epsilon ||x||\}.$$

The corresponding order topology τ_o is the topology induced by $||\cdot||$.

Order topology

Theorem

If X there are non empty, order open and order bounded sets in X, then τ_o is a linear topology if and only if X is Archimedean and directed.

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Let X be an Archimedean vector lattice. Then the following statements are equivalent:

(i) *X has non empty, order open and order bounded sets.*

(ii) *X_+ has oder topological interior points.*

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(i) *X has non empty, order open and order bounded sets.*

(ii) *X_+ has oder topological interior points.*

(iii) *X is finite dimensional*

Order convergence

Definition

Let $x \in P$ and let $(x_\alpha)_{\alpha \in A}$ be a net in P. We define

(i) $x_\alpha \xrightarrow{o_1} x$, if there are nets $(\hat{x}_\alpha)_{\alpha \in A}$ and $(\check{x}_\alpha)_{\alpha \in A}$ in P such that $\check{x}_\alpha \downarrow x$, $\hat{x}_\alpha \uparrow x$ and $\hat{x}_\alpha \leq x_\alpha \leq \check{x}_\alpha$ for every $\alpha \in A$.

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(ii) $x_\alpha \xrightarrow{o_2} x$, if there are nets $(\hat{x}_\alpha)_{\alpha \in A}$ and $(\check{x}_\alpha)_{\alpha \in A}$ in P and $\alpha_0 \in A$ such that $\check{x}_\alpha \downarrow x$, $\hat{x}_\alpha \uparrow x$ and $\hat{x}_\alpha \leq x_\alpha \leq \check{x}_\alpha$ for every $\alpha \in A_{\geq \alpha_0}$.

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(ii) $x_\alpha \xrightarrow{o_2} x$, if there are nets $(\hat{x}_\alpha)_{\alpha \in A}$ and $(\check{x}_\alpha)_{\alpha \in A}$ in P and $\alpha_0 \in A$ such that $\check{x}_\alpha \downarrow x$, $\hat{x}_\alpha \uparrow x$ and $\hat{x}_\alpha \leq x_\alpha \leq \check{x}_\alpha$ for every $\alpha \in A_{\geq \alpha_0}$.

(iii) $x_\alpha \xrightarrow{o_3} x$, if there are nets $(\hat{x}_\beta)_{\beta \in B}$ and $(\check{x}_\gamma)_{\gamma \in C}$ in P and a map $\eta \colon B \times C \to A$ such that $\hat{x}_\beta \uparrow x$, $\check{x}_\gamma \downarrow x$ and $\hat{x}_\beta \leq x_\alpha \leq \check{x}_\gamma$ for every $\beta \in B$, $\gamma \in C$ and $\alpha \in A_{\geq \eta(\beta,\gamma)}$.

Order convergence

Proposition

Let $x \in P$ and let $(x_\alpha)_{\alpha \in A}$ be a net in P. Then

(i) $x_\alpha \xrightarrow{o_1} x$ implies $x_\alpha \xrightarrow{o_2} x$,

(ii) $x_\alpha \xrightarrow{o_2} x$ implies $x_\alpha \xrightarrow{o_3} x$, and

(iii) $x_\alpha \xrightarrow{o_3} x$ implies $x_\alpha \xrightarrow{\tau_o} x$.

Order convergence

Example ($x_\alpha \xrightarrow{o_2} x$ does not imply $x_\alpha \xrightarrow{o_1} x$)

Let $P = \mathbb{R}$. Equip $A := \mathbb{R}_{\geq 0}$ with the following ordering: Let $A \setminus \{0\}$ be ordered as in P. Let $a \leq 0$ for $a \in A \setminus \{0\}$. Set $x_\alpha := \alpha$ for $\alpha \in A$. Then $x_\alpha \xrightarrow{o_2} x$ but not $x_\alpha \xrightarrow{o_1} x$.

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Example by Fremlin in [AbSi05, Example 1.4].

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Theorem

Let P be a Dedekind complete lattice, let $(x_\alpha)_{\alpha \in A}$ be a net in P and $x \in P$. Then $x_\alpha \xrightarrow{o_2} x$ if and only if $x_\alpha \xrightarrow{o_3} x$.

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Example ($x_\alpha \xrightarrow{\tau_o} x$ does not imply $x_\alpha \xrightarrow{o_3} x$)

Let X be the vector lattice of all real, Lebesgue-measurable, almost everywhere finite functions on $[0, 1]$. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of characteristic functions of the intervals

$$[0, 1], [0, \tfrac{1}{2}], [\tfrac{1}{2}, 1], [0, \tfrac{1}{4}], [\tfrac{1}{4}, \tfrac{2}{4}], [\tfrac{2}{4}, \tfrac{3}{4}], [\tfrac{3}{4}, 1], [0, \tfrac{1}{8}], \ldots$$

The sequence $(f_n)_{n\in\mathbb{N}}$ does not o_3-converge but τ_o-convergences to 0.

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The sequence $(f_n)_{n\in\mathbb{N}}$ does not o_3-converge but τ_o-convergences to 0.

Theorem (Imhoff)

If X is a Archimedean and directed and contains non empty order open and order bounded sets, then for every net $(x_\alpha)_{\alpha\in A}$ in X and every $x \in X$ we have $x_\alpha \xrightarrow{o_3} x$ if and only if $x_\alpha \xrightarrow{\tau_o} x$.

Theorem

Let $i \in \{1, 2, 3\}$ and $C \subseteq P$. The following statements are equivalent:

(i) *C is order closed.*

(ii) *For every net $(x_\alpha)_{\alpha \in A}$ in C with $x_\alpha \xrightarrow{o_i} x \in P$ it follows that $x \in C$.*

Order continuity

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(i) *o_i-continuous in $x \in P$*, if for every net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \xrightarrow{o_i} x$ we have that $f(x_\alpha) \xrightarrow{o_i} f(x)$ (where $i \in \{1, 2, 3\}$).

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f is called *o_i-continuous* (*order continuous*, respectively) if it is o_i-continuous (order continuous, respectively) in x for every $x \in P$.

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Theorem

Let $i \in \{1, 2, 3\}$. Every o_i-continuous map $f\colon P \to Q$ is order continuous.

Order continuity

Theorem

Let $f\colon P \to Q$ be a monotone map and $i \in \{1,2,3\}$. Then the following statements are equivalent:

(i) *f is o_i-continuous.*

(ii) *f is order continuous.*

Order continuity

Theorem

Let $f\colon P \to Q$ be a monotone map and $i \in \{1,2,3\}$. Then the following statements are equivalent:

(i) *f is o_i-continuous.*

(ii) *f is order continuous.*

(iii) *For every net $(x_\alpha)_{\alpha\in A}$ in P and $x \in P$ the following implications are valid:*

   (a) *If $x_\alpha \downarrow x$ then $\inf\{f(x_\alpha); \alpha \in A\}$ exists and satisfies $\inf\{f(x_\alpha); \alpha \in A\} = f(x)$.*

   (b) *If $x_\alpha \uparrow x$ then $\sup\{f(x_\alpha); \alpha \in A\}$ exists and satisfies $\sup\{f(x_\alpha); \alpha \in A\} = f(x)$.*

Order continuity

Corollary

Every order embedding $f : P \to Q$ for which $f[P]$ is order dense in Q is order continuous (and, hence, o_i-continuous, where $i \in \{1, 2, 3\}$).

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Definition

Let P and Q be partially ordered sets and $f\colon P \to Q$ a map. f is called an *order embedding*, if for every $x, y \in P$ there holds $x \leq y$ if and only if $f(x) \leq f(y)$.

Order continuity

Corollary

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Definition

We call $M \subseteq P$ *order dense* in P if for every $x \in P$ one has

$$\sup M_{\leq x} = x = \inf M_{\geq x}.$$

Partially ordered abelian groups

Definition

$(G, +, 0)$ *partially ordered abelian group* $:\Leftrightarrow$ G abelian group with a partial order such that for every $x, y, z \in G$ with $x \leq y$ it follows $x + z \leq y + z$.

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G is *Archimedean* if for every $x, y \in G$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$.

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Definition

G is *Archimedean* if for every $x, y \in G$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$.
G has the *Riesz decomposition property* if for every $x, y \in G_+$ and $w \in [0, x + y]$ there are $u \in [0, x]$ and $v \in [0, y]$ such that $w = u + v$.

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G *lattice-ordered abelian group*, if G is a lattice.

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$\mathrm{A}_+(G, H) :=$ set of all additive and positive maps from G to H

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If G is directed, then $\mathrm{A}(G, H)$ and $\mathrm{A}_\mathrm{b}(G, H)$ are partially ordered abelian groups:

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$\mathrm{A}_\mathrm{b}(G,H) :=$ set of all additive and order bounded maps from G to H
If G is directed, then $\mathrm{A}(G,H)$ and $\mathrm{A}_\mathrm{b}(G,H)$ are partially ordered abelian groups: $f+g\colon G \to H$, $x \mapsto f(x)+g(x)$; $0\colon x \mapsto 0$; $f \leq g$ whenever for every $x \in G_+$ we have $f(x) \leq g(x)$.

Partially ordered abelian groups

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$\mathrm{A}_{\mathrm{b}}^{\tau_o}(G,H) :=$ set of all additive, order bounded and order continuous maps from G to H
$\mathrm{A}_{+}^{\mathrm{oc}}(G,H) :=$ set of all additive, positive and order continuous operators.

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$\mathrm{A}_{+}^{\mathrm{oc}}(G,H) :=$ set of all additive, positive and order continuous operators.

Theorem

The following statements are equivalent for $f : G \to H$.

(i) $f \in \mathrm{A}_{+}^{\mathrm{oc}}(G,H)$

(ii) *f additive, positive and o_i-continuous operators for some (any) $i \in \{1,2,3\}$.*

(iii) *for every net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ it holds $f(x_\alpha) \downarrow 0$.*

Partially ordered abelian groups

Theorem

Let G be a directed partially ordered abelian group that satisfies the Riesz decomposition property and let H be a Dedekind complete lattice-ordered abelian group. Then

$$\begin{aligned} \mathrm{A}_{\mathrm{b}}^{\mathcal{O}_1}(G,H) &= \mathrm{A}_{\mathrm{b}}^{\mathcal{O}_2}(G,H) = \mathrm{A}_{\mathrm{b}}^{\mathcal{O}_3}(G,H) = \mathrm{A}_{\mathrm{b}}^{\mathcal{T}_o}(G,H) \\ &= \mathrm{A}_{+}^{\mathrm{oc}}(G,H) - \mathrm{A}_{+}^{\mathrm{oc}}(G,H). \end{aligned}$$

is an order closed ideal in $\mathrm{A}_{\mathrm{b}}(G,H)$.

Definition

A directed subgroup I of G is called an *ideal*, if for every $x, y \in I$ one has $[x,y] \subseteq I$.

Partially ordered vector spaces

X and Y ... partially ordered vector spaces

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X and Y ... partially ordered vector spaces

Proposition

If X is directed and Y is Archimedean, then every additive monotone map is linear, i.e. $\mathrm{A}_+(X, Y) = \mathrm{L}_+(X, Y)$.

Partially ordered vector spaces

X and Y ... partially ordered vector spaces

Proposition

If X is directed and Y is Archimedean, then every additive monotone map is linear, i.e. $\mathrm{A}_+(X, Y) = \mathrm{L}_+(X, Y)$.

Proposition

Let X, Y be directed and Archimedean and let $i \in \{1, 2, 3\}$. Then every additive o_i-continuous map from X to Y is linear, hence $\mathrm{A}_{\mathrm{b}}^{o_i}(X, Y) = \mathrm{L}_{\mathrm{b}}^{o_i}(X, Y)$.

Partially ordered vector spaces

X and Y ... partially ordered vector spaces

Proposition

If X is directed and Y is Archimedean, then every additive monotone map is linear, i.e. $\mathrm{A}_+(X, Y) = \mathrm{L}_+(X, Y)$.

Proposition

Let X, Y be directed and Archimedean and let $i \in \{1, 2, 3\}$. Then every additive o_i-continuous map from X to Y is linear, hence $\mathrm{A}_{\mathrm{b}}^{o_i}(X, Y) = \mathrm{L}_{\mathrm{b}}^{o_i}(X, Y)$. Furthermore, $\mathrm{A}_+^{\mathrm{oc}}(X, Y) = \mathrm{L}_+^{\mathrm{oc}}(X, Y)$.

Partially ordered vector spaces

X and Y ... partially ordered vector spaces

Proposition

If X is directed and Y is Archimedean, then every additive monotone map is linear, i.e. $\mathrm{A}_+(X, Y) = \mathrm{L}_+(X, Y)$.

Proposition

Let X, Y be directed and Archimedean and let $i \in \{1, 2, 3\}$. Then every additive o_i-continuous map from X to Y is linear, hence $\mathrm{A}_\mathrm{b}^{o_i}(X, Y) = \mathrm{L}_\mathrm{b}^{o_i}(X, Y)$. Furthermore, $\mathrm{A}_+^{\mathrm{oc}}(X, Y) = \mathrm{L}_+^{\mathrm{oc}}(X, Y)$.

Proposition

Let X be a directed partially ordered vector space with the Riesz decomposition property, and let Y be a Dedekind complete vector lattice. Then every additive order bounded map is homogeneous, i.e. $\mathrm{A}_\mathrm{b}(X, Y) = \mathrm{L}_\mathrm{b}(X, Y)$.

Theorem

Let X be a directed partially ordered vector space with the Riesz decomposition property, and let Y be a Dedekind complete vector lattice. Then

$$\begin{aligned} \mathrm{L}_{\mathrm{b}}^{o_1}(X,Y) = \mathrm{L}_{\mathrm{b}}^{o_2}(X,Y) = \mathrm{L}_{\mathrm{b}}^{o_3}(X,Y) = \mathrm{L}_{\mathrm{b}}^{\tau_o}(X,Y) \\ = \mathrm{L}_{+}^{\mathrm{oc}}(X,Y) - \mathrm{L}_{+}^{\mathrm{oc}}(X,Y). \end{aligned}$$

is an order closed ideal in $\mathrm{L}_{\mathrm{b}}(X,Y)$.

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