Some applications of vector lattices on the axiomatic theory of risk measures

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Coherent risk measures

In the milestone paper (Coherent measures of risk, P. Artzner, et al. , **Math. Fin.**, 1999) that has **9313** citations (Google Scholar) the authors establish an axiomatic theory of risk measures.
In the following by $\mathcal{X}$ we will denote an **ideal** of $L^1(\Omega, \Sigma, \mathbb{P})$ such that $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathcal{X} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Definition

A mapping $\rho : \mathcal{X} \to (-\infty, \infty]$ is called a coherent risk measure if it satisfies the following conditions.

- (Monotonicity) If $X \leq Y$ then $\rho(X) \geq \rho(Y)$.
- (Positive homogeneity) If $\lambda \geq 0$ then $\rho(\lambda X) = \lambda \rho(X)$.
- (Subadditivity) $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- (Cash invariance) If $m \in \mathbb{R}$ then $\rho(X + m) = \rho(X) - m$.

Acceptance sets

Definition

Let $\mathcal{X}$ be an ideal of $L^1(\mathbb{P})$. A non-empty set $\mathcal{C}$ in $\mathcal{X}$ is said to be an acceptance set whenever

(i) $\mathcal{C}$ is *monotone* (i.e. $X_1 \geq X_2 \in \mathcal{C}$ implies $X_1 \in \mathcal{C}$),

(ii) $\mathcal{C}$ is *positively homogeneous* (i.e. $\lambda\mathcal{C} \subset \mathcal{C}$ for any $\lambda \geq 0$),

(iii) $\mathcal{C}$ is *additive* (i.e. $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$).

(iv) For any $X \in \mathcal{X}$, there exists $m \in \mathbb{R}$ so that $X + m\mathbf{1} \notin \mathcal{C}$.

Remark

(i) *If $\rho : \mathcal{X} \to (-\infty, \infty]$ is a coherent risk measure then the set $\mathcal{C} = \{X \in \mathcal{X} \mid \rho(X) \leq 0\}$ is an acceptance set.*

(ii) *If $\mathcal{C}$ is an acceptance set in $\mathcal{X}$, then the following mapping defines a coherent risk measure on $\mathcal{X}$.*

$$\rho_{\mathcal{C}}(X) = \inf\{m \in \mathbb{R} : X + m\mathbf{1} \in \mathcal{C}\}.$$

Dual representations of convex functionals in LCS

Definition

Let $(\mathcal{X}, \tau)$ be a locally convex space and $\rho : \mathcal{X} \to (-\infty, \infty]$ a functional on $\mathcal{X}$. We say that ρ is lower-semicontinuous whenever the sublevel set $\mathcal{C}_\lambda = \{X \in \mathcal{X} \mid \rho(X) \leq \lambda\}$ is τ-closed for all $\lambda \in \mathbb{R}$.

Theorem (Fenchel-Moreau Dual Representation)

Let $(\mathcal{X}, \tau)$ be a locally convex topological space and $\rho : \mathcal{X} \to (-\infty, \infty]$ a convex functional. Then the following are equivalent

(i) *ρ is lower-semicontinuous.*

(ii) *ρ admits the following representation*

$$\rho(X) = \sup_{Y \in \mathcal{X}^*} \{\langle X, Y \rangle - \rho^*(Y)\}$$

where $\rho^*(Y) = \sup_{X \in \mathcal{X}} \{\langle X, Y \rangle - \rho(X)\}$

The model space $\mathcal{X}$

The standard theory of risk measures is developed for the case where $\mathcal{X} = L^\infty(\mathbb{P})$. The mathematical toolbox in this framework is quite rich, however the model space itself is quite small (e.g. normal random variables are not in $\mathcal{X}$).
Orlicz spaces arose as a more general model space $\mathcal{X}$.

1. Cheridito and Li, ‘09
2. Biagini and Frittelli, ‘10
3. Orihuela and Ruiz Galán, ‘12
4. Krätschmer, Schied and Zähle, ‘14
5. Arai, Fukasawa ’14
6. Delbaen and Owari, ’16
7. Bellini, Laeven and Rosazza Gianin, ’17

Orlicz spaces

Throughout this talk, (Φ, Ψ) stands for an **Orlicz pair**, where

- $\Phi : [0, \infty) \to [0, \infty)$ is convex, increasing,
- $\Phi(0) = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$
- $\Psi(s) = \sup\{ts - \Phi(t) : t \geq 0\}$. for all $s \geq 0$.

The **Orlicz space** $L^\Phi := L^\Phi(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all real-valued random variables X (modulo a.s. equality) such that

$$\|X\|_\Phi := \inf\left\{\lambda > 0 : \mathbb{E}\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] \leq 1\right\} < \infty.$$

The subspace of L^Φ consisting of all $X \in L^\Phi$ such that

$$\mathbb{E}\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] < \infty \quad \text{for all } \lambda > 0$$

is the **Orlicz heart** of L^Φ and is denoted by H^Φ.

Δ_2-condition

Definition

An Orlicz function Φ is said to satisfy the Δ_2 condition, if there exist $t_0 \in (0, \infty)$ and $k \in \mathbb{R}$ such that $\Phi(2t) < k\Phi(t)$ for all $t \geq t_0$.

Remark

Φ satisfies the Δ_2 condition if and only if $L^\Phi = H^\Phi$.

Example

- For $\Phi(x) = \frac{x^p}{p}, p > 1$ we have that $L^\Phi = H^\Phi = L^p$.
- For $\Phi(x) = \exp(x) - x - 1$ we have that $\{0\} \neq H^\Phi \neq L^\Phi$.

The aim of the talk

Theorem (Delbaen '02)

Let $\rho : L^\infty(\mathbb{P}) \to (-\infty, \infty]$ be a coherent risk measure. Then the following are equivalent

(i) *ρ has the Fatou property (i.e. $\rho(X) \leq \liminf_n \rho(X_n)$ whenever $X_n \to X$ a.s. and (X_n) is bounded in L^∞).*

(ii) *There exists a set $\mathcal{Q}$ of probabilities in $\mathcal{D} = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}\}$ such that*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(-X)$$

In this talk we will discuss possible extensions of the above result in the case where $L^\infty(\mathbb{P})$ is replaced by a general model space $\mathcal{X}$.

The aim of the talk

Theorem (Delbaen '02)

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In this talk we will discuss possible extensions of the above result in the case where $L^\infty(\mathbb{P})$ is replaced by an Orlicz space $L^\Phi(\mathbb{P})$.

Fatou property and order closedness

Let $\mathcal{X}$ be an ideal of $L^0(\mathbb{P})$ and (X_n) a sequence in $\mathcal{X}$.
We will write $X_n \xrightarrow{o} X$ for some $X \in \mathcal{X}$, whenever

$$X_n \xrightarrow{a.s} X \in \mathcal{X}, \;\; |X_n| \leq Y \text{ for some } Y \in \mathcal{X}$$

A set $\mathcal{C}$ in $\mathcal{X}$ is said to be **order closed** whenever it contains the limit of every order convergent sequence with elements in $\mathcal{C}$.

Definition (Biagini & Frittelli '09)

A coherent risk measure $\rho : \mathcal{X} \to (-\infty, \infty]$ is said to have the *Fatou property* if the sublevel set $\mathcal{C}_\lambda = \{X \in \mathcal{X} \mid \rho(X) \leq \lambda\}$ is order closed for all $\lambda \in \mathbb{R}$.

Order closed convex sets in L^∞

Theorem (Grothendieck)

Let $\mathcal{C}$ be a convex set in L^∞. Then the following are equivalent.

(i) *$\mathcal{C}$ is $\sigma(L^\infty, L^1)$-closed.*

(ii) *$\mathcal{C}$ is order closed.*

In view of the above result, we have that a coherent risk measure on L^∞ has the Fatou property precisely when it is $\sigma(L^\infty, L^1)$-lower semicontinuous and Delbaen's representations Theorem can be deduced by applying the Fenchel-Moreau Dual Representation.

L^Ψ and H^Ψ representations of risk measures

$$\mathcal{D}_{L^\Psi} = \{\text{all probabilities } \mathbb{Q} \text{ such that } \tfrac{d\,\mathbb{Q}}{d\,\mathbb{P}} \in L^\Psi\}$$
$$\mathcal{D}_{H^\Psi} = \{\text{all probabilities } \mathbb{Q} \text{ such that } \tfrac{d\,\mathbb{Q}}{d\,\mathbb{P}} \in H^\Psi\}$$

Definition

We say that a coherent risk measure $\rho : L^\Phi \to (-\infty, \infty]$ admits a representation via $L^\Psi (H^\Psi)$ whenever

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(-X),$$

where $\mathcal{Q}$ is a set of probabilities in $\mathcal{D}_{L^\Psi} (\mathcal{D}_{H^\Psi})$

Remark

If ρ admits a representation via L^Ψ or H^Ψ, then ρ has the Fatou property.

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Order closed convex sets in L^Φ

To extend Delbaen's representation Theorem to Orlicz spaces one has to solve the following problem.

Problem (Owari '14)

Is every order closed convex set in L^Φ always $\sigma(L^\Phi, L^\Psi)$-closed?

Theorem (Cheridito & Li '09)

Suppose that Φ satisfies the Δ_2-condition. Then every order closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.

The false claim...

Definition (Biagini & Frittelli '09)

The topology $\sigma(L^\Phi, L^\Psi)$ is said to have the C-property if for any convex set $\mathcal{C}$ in L^Φ and $X \in \overline{\mathcal{C}}^{\sigma(L^\Phi, L^\Psi)}$ there exists a sequence (X_n) in $\mathcal{C}$ such that $X_n \xrightarrow{o} X$.

In the quite popular paper of Biagini & Frittelli it was claimed that $\sigma(L^\Phi, L^\Psi)$ has the C-property for any Orlicz pair (Φ, Ψ) from which it is immediate that Delbaen's representation theorem can be extended to any Orlicz space.

This assertion turned out to be false...

Theorem (GX)

$\sigma(L^\Phi, L^\Psi)$ has the C-property if and only if Φ satisfies the Δ_2-condition.

Bounded order closed convex sets in L^Φ

Theorem (GLX)

Denote by $\mathcal{B}$ the closed unit ball in L^Φ. For an order closed convex set $\mathcal{C}$ in L^Φ, we have that $\mathcal{C} \cap k\mathcal{B}$ is $\sigma(L^\Phi, L^\Psi)$-closed for all $k \geq 1$.

Theorem (Krein-Smulian)

Let $\mathcal{X}$ be a Banach space and $\mathcal{C}$ a convex set in $\mathcal{X}^*$ such that $\mathcal{C} \cap k\mathcal{B}$ is w^*-closed for all $k \geq 1$, where $\mathcal{B}$ is the closed unit ball of $\mathcal{X}^*$. Then $\mathcal{C}$ is w^*-closed.

If Ψ satisfies the Δ_2-condition, then $\sigma(L^\Phi, L^\Psi)$ is a w^*-topology and thus we get that

Corollary (GLX)

If Ψ satisfies the Δ_2-condition, then every order closed convex set in L^ϕ is $\sigma(L^\Phi, L^\Psi)$-closed.

In conclusion, when either Φ or Ψ satisfies the Δ_2-condition we get that any coherent risk measure on L^Φ has the Fatou property iff it admits a dual representation via L^Ψ.

When the representation fails...

Theorem (GLX)

If Φ and Ψ both fail the Δ_2-condition, then L^Φ admits an acceptance subset $\mathcal{C}$ which is order closed but not $\sigma(L^\Phi, L^\Psi)$-closed. In particular $\rho_\mathcal{C}$ is a coherent risk measure on L^Φ with the Fatou property that do not admit a representation via L^Ψ.

Example

Consider the function

$$\phi(t) = \begin{cases} t & 0 \leq t < 1 \\ n! & (n-1)! \leq t < n!, n \geq 2 \end{cases}$$

Then the Orlicz function $\Phi(x) = \int_0^x \phi(t)dt$ fails the Δ_2-condition. In particular, $\Phi(2 \cdot n!) > n\Phi(n!)$. The same holds for its conjugate $\Psi(x) = \int_0^x \psi(t)dt$, where

$$\psi(t) = \begin{cases} t & 0 \leq t < 1 \\ (n-1)! & (n-1)! \leq t < n!, n \geq 2 \end{cases}$$

A variant of the C-property

Theorem (Biagini & Frittelli)

Suppose that Φ satisfies the Δ_2-condition. Then $\sigma(L^\Phi, L^\Psi)$ has the C-property. That is for any convex set $\mathcal{C}$ in L^Φ and $X \in \overline{\mathcal{C}}^{\sigma(L^\Phi, L^\Psi)}$ there exists a sequence (X_n) such that $X_n \xrightarrow{o} X$.

Lemma (GX)

For any convex set in $\mathcal{C}$ and $X \in \overline{\mathcal{C}}^{\sigma(L^\Phi, H^\Psi)}$ there exists a sequence (X_n) in $\mathcal{C}$ such that $X_n \xrightarrow{a.s.} X$.

The above Lemma lead us to the definition of the Strong Fatou property.

Strong Fatou property

Definition

A functional $\rho : L^\Phi \to (-\infty, \infty]$ has the strong Fatou property if $\rho(X) \leq \liminf_n \rho(X_n)$, whenever (X_n) is norm bounded in L^Φ and $X_n \xrightarrow{a.s.} X$.

Theorem (GX)

Let $\rho : L^\Phi(\mathbb{P}) \to (-\infty, \infty]$ be a coherent risk measure. Then the following are equivalent

(i) *ρ has the strong Fatou property.*

(ii) *ρ admits a representation via H^Ψ. In particular we have*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(-X)$$

where $\mathcal{Q}$ is a set of probabilities in $\mathcal{D}_{H_\Psi}$.

Law invariant risk measures

Definition

- A functional $\rho : \mathcal{X} \to (-\infty, \infty]$ is said to be law invariant whenever $\rho(X) = \rho(Y)$ for each $X, Y \in \mathcal{X}$ with $X \overset{\mathbb{P}}{\sim} Y$.
- A set $\mathcal{C}$ in $\mathcal{X}$ is law-invariant whenever if $X \in \mathcal{C}$ for any $X \in \mathcal{X}$ that has the same law of some element of C.

In the sequel, we will write π to denote a finite measurable partition of Ω whose members all have non-zero probabilities, and write $\sigma(\pi)$ to denote the finite σ-subalgebra generated by π.

Theorem (CGX)

(i) *For any $X \in \mathcal{X}$ there exists a sequence (π_n) such that*

$$\mathbb{E}[X|\sigma(\pi_n)] \xrightarrow{o} X.$$

(ii) *Let $\mathcal{C}$ be a convex, law-invariant, order closed set in $\mathcal{X}$. Then, for every $X \in \mathcal{X}$ we have $X \in \mathcal{C}$ if, and only if,*

$$\mathbb{E}[X|\sigma(\pi)] \in C \ \forall \ \pi$$

Representations under law-invariance

Theorem (RX)

Let $\rho : \mathcal{X} \to (-\infty, \infty]$ be a law-invariant coherent risk measure. TFAE:

(i) *ρ has the Fatou property*

(ii) *ρ admits a representation via $\mathcal{D}_{L_\infty}$*

If moreover, $\mathcal{X}$ is a Banach lattice different than L^1, then (i)-(ii) are equivalent to the following

(iii) *ρ has the strong Fatou property*

Sketch proof.

$(i) \Rightarrow (ii)$: Let $\mathcal{C}$ be a sublevel set of ρ. Then by (i) we have that $\mathcal{C}$ is a convex, law-invariant order closed set. Therefore $\mathbb{E}(Y \mid \sigma(\pi)) \in \mathcal{C}$ for each $Y \in \mathcal{C}$ and π.

Fix a net $(X_\alpha) \subset C$ and $X \in \mathcal{X}$ such that $X_\alpha \xrightarrow{\sigma(\mathcal{X}, L^\infty)} X$. Then for any π we have that $\mathbb{E}(X_\alpha|\sigma(\pi)) \xrightarrow{o} \mathbb{E}(X|\sigma(\pi))$. Since $\mathcal{C}$ is also order closed we conclude that $\mathbb{E}(X|\sigma(\pi)) \in \mathcal{C}$. Finally, by picking a sequence (π_n) such that $\mathbb{E}[X|\sigma(\pi_n)] \xrightarrow{o} X$. we get that $X \in \mathcal{C}$.

□

A concrete example

Fix a finite-valued Orlicz function Φ that is normalized by $\Phi(1) = 1$. For a given $\alpha \in (0,1)$ we consider the Orlicz function $\Phi_\alpha := \frac{\Phi}{1-\alpha}$.

Definition

The *Haezendonck-Goovaerts risk measure* associated to Φ at level α is the map $\rho_\alpha : L^\Phi \to \mathbb{R}$ defined by

$$\rho_\alpha(X) := \inf_{m \in \mathbb{R}} \{ m + ||(-X-m)^+||_{\Phi_\alpha} \}$$

In GMX we proved that ρ_α has the Fatou property and thus by applying our results for law-invaraint coherent risk measures we were able to derive the following representation.

Theorem (GMX)

The Haezendonck-Goovaerts risk measure ρ_α satisfies

$$\rho_\alpha(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_\alpha^\infty} \mathbb{E}_\mathbb{Q}[-X]$$

for every $X \in L^\Phi$, where $\mathcal{Q}_\alpha^\infty = \left\{ \mathbb{Q} \in \mathcal{P}(\mathbb{P}) ; \; \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty, \; \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^{\Psi_\alpha}} \leq 1 \right\}$.

The last(?) piece of the puzzle

From the celebrated work of Jouini, Schachermayer and Touzi (2006) it follows that any law invariant real-valued coherent risk measures on L^∞ have automatically the Fatou property.

Is this true for any model space $\mathcal{X}$?

Thanks for your attention!